Comparisons of conservative and nonconservative results were made with all aspects of the computations identical except the finite-differencing scheme. In all cases the calculations were iterated until the root-mean-square value of the true residual was of the order of the truncation error. Although the solution itself was observed to change somewhat for a given case when the analytical stretching was changed, it is felt that the relative streamline shape effects presented here indicate that a nonconservative finite-difference scheme destroys the global mass balance in a supercritical flow calculation when shocks are present.

Results

Comparison cases were run for a 10% thick parabolic arc airfoil at zero incidence for freestream Mach numbers of 0., 0.70, 0.84, and 0.95. These represent incompressible, subcritical, mildly supercritical, and strongly supercritical flow conditions, respectively. The computational grid was 128 streamwise by 33 in the normal direction (the physical half-space, airfoil mean plane to infinity). Streamline deflections were computed along several grid lines for all cases; only three are shown in the figures.

The incompressible ($M=0$) and subcritical ($M=0.70$) results for pressure distributions and streamline deflections were identical for conservative and nonconservative finite differencing. In all cases the computed streamtubes returned to their proper size. Since the conservative and nonconservative schemes differ only at points where the flow is supersonic, the results were expected to agree.

Figure 1 shows the streamwise distribution of pressure coefficients along the symmetry line $y=0$ for both the mildly ($M=0.84$) and strongly ($M=0.95$) supercritical flows. Conservative results are given by the solid curves while nonconservative results are given by dashed curves. As others have shown in the past, one effect of the conservative scheme is to locate the shock wave further downstream on the airfoil surface. Figure 1a shows that the mildly supercritical flows are similar. It can be seen in Fig. 1b, however, that the strongly supercritical flows are no longer similar. In the nonconservative case there is a normal shock at the airfoil trailing edge whereas conservative differencing gives a weak oblique shock wave at the trailing edge followed by a normal shock located about 1/2 a chord length downstream of the airfoil.

Computed streamline deflections (from straight lines) $\Delta y/c$ for both mildly and strongly supercritical flows are shown in Fig. 2. Note that the scale of the ordinate ($\Delta y/c$) has been magnified 20 times that of the abcissa (x/c) for clarity. Tick marks at the edges of the figure show asymptotic values. The zero levels of streamline deflection (i.e., $\Delta y/c$ at upstream infinity) for three different streamlines are given by the tick mark at the left edge. The deflection curves are labeled with the upstream infinity value of y/c for the streamline itself. Tick marks at the right edge of the figure denote the levels of streamline deflection $\Delta y/c$ at downstream infinity; numerical values are also indicated. The results for mildly ($M=0.84$) supercritical flow are shown in Fig. 2a. Observe that the streamline deflections at downstream infinity show the conservative streamtube sizes return to their upstream infinity values whereas the nonconservative ones do not. For this case, the shock wave extends about 1/2 a chord length into the flow. Similar results are shown for the strongly ($M=0.95$) supercritical flow in Fig. 2b. Here it is very evident that the nonconservative streamline deflections do not return to zero far downstream of the airfoil. On the other hand, however, the conservative streamtubes are seen to return to their proper size far downstream. For this case, the shock waves extend several chord lengths into the flow.

Use of a nonconservative finite-difference scheme in transonic flow calculations destroys the global mass balance when shocks are present. The mass created by the nonconservative operator at the shock produces a nonphysical swelling of the inviscid streamtubes which persists far downstream. Perhaps the fortuitous agreement between the nonconservative and experimental results comes about because this streamtube swelling effect simulates a viscous wake or thickened boundary layer downstream of a shockwave. In any case, it is felt that the conservative finite-difference scheme should be used in applications where the streamtube effects are important, such as internal or confined flows.

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Technical Comments

Comment on "Rapid Finite-Difference Computation of Subsonic and Slightly Supercritical Aerodynamic Flows"

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At the end of their paper,[1] Martin and Lomax state that they consider their main contribution to be the use of the Aitken/Shanks extrapolation formula

$$u^* = (u_1 u_3 - u_2^2)/(u_1 - 2u_2 + u_3) \tag{1}$$

to accelerate iterative convergence. This formula is well known to produce very useful results when properly applied; it can also produce indifferent or poor results when used inappropriately. My main purpose in writing this comment is to emphasize the conditions under which this formula can be used to extrapolate to the sum of a series. A secondary purpose is to give some additional discussion of the "one-dimensional example."

What was shown by Aitken, Shanks, and others is that the terms of a convergent power series, and of many divergent series, eventually behave like the terms of a geometric series to within some acceptable accuracy, at which point the extrapolation formula [Eq. (1)] can be applied to sum the tail of the series. This is not quite the same as applying the for-

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mula directly to the first three terms of the series without any a priori proof or a posteriori evidence that the series in question is very nearly a geometric series right from the start.

To help clarify my point, I give a derivation of the Aitken/Shanks formula, emphasizing the condition which must be met for its successful application, and then show how it is properly used to sum a power series.

Finally I take up the "one-dimensional example" of Martin and Lomax and show that more can be squeezed out of it than they tried to do. Actually, it is a pedagogically desirable "transparent example" in that every step of their procedure can be exhibited explicitly and the procedure yields the known exact solution. At the end I draw attention to other recent work related to the use of rational functions to extrapolate to the limit of a sequence.

Let us consider a sequence of real numbers $s_1, s_2, \ldots, s_n, s_{n+1}, \ldots$ which, from some value n_0 of n on, satisfy the condition

$$s_{n+2} - s_{n+1} = a(s_{n+1} - s_n) \quad (n \geq n_0) \tag{2}$$

where a is a constant and

$$a \neq 1 \tag{3}$$

This constancy of the ratio of successive differences is the property which characterizes the partial sums of the geometric series and which we now proceed to use. For any $n > n_0$ we construct

$$s_n^* = \lim_{p \to \infty} s_{n+p} = s_n + (s_{n+1} - s_n) + (s_{n+2} - s_{n+1}) + \ldots \tag{4}$$

(This limit has meaning if the sequence s_n converges; otherwise this is just formal manipulation and the result is called an anti-limit.) Making use of Eq. (2), we have

$$s_n^* = s_n + (s_{n+1} - s_n)(1 + a + a^2 + \ldots) \tag{5}$$

$$= s_n + \frac{s_{n+1} - s_n}{1 - a} \tag{6}$$

$$= s_n + \frac{s_{n+1} - s_n}{1 - \dfrac{s_{n+2} - s_{n+1}}{s_{n+1} - s_n}}$$

$$= s_n - \frac{(s_{n+1} - s_n)^2}{s_n - 2s_{n+1} + s_{n+2}} \tag{7}$$

$$= \frac{s_n s_{n+2} - s_{n+1}^2}{s_n - 2s_{n+1} + s_{n+2}} \tag{8}$$

Thus we have drived the Aitken/Shanks formula Eq. (1), in a manner which shows clearly why Eq. (2) must be satisfied, at least to some specified degree of accuracy, from some point in the sequence onward. The step from Eqs. (5) and (6) (summation of the geometric series) is a formal step made regardless of whether the series converges ($|a| < 1$) or diverges ($|a| > 1$).

The standard method of using Eq. (8) on a sequence which is already given (such as the sequence obtained from a power series) is to calculate the extrapolated sequence $s_0^*, s_1^*, s_2^*, \ldots$ out to some point where s_n^* and s_{n+1}^* agree to the desired number of decimal places. This definitely need not take place right at the beginning of the extrapolated sequence. In case the extrapolated sequence converges too slowly, it can be used as input into the same process to produce a second extrapolated sequence $s_0^{**}, s_1^{**}, s_2^{**}$, etc. According to Shanks[2] this was used by Aitken[3] in his first publication on the subject.

Let us look more closely at the case where the s_n are the partial sums of a power series:

$$s_n = \sum_{k=0}^{n-1} c_k t^k \tag{9}$$

We have

$$s_{n+1} - s_n = c_n t^n \tag{10}$$

and the condition [Eq. (2)] for applicability of the extrapolation process yields

$$\frac{c_{n+1} t^{n+1}}{c_n t^n} = \frac{c_{n+1}}{c_n} t = a \ t = \text{constant} \tag{11}$$

(Here t is regarded as a constant.) This condition is met approximately by many familiar power series for sufficiently large values of n.

The power series employed by Martin and Lomax has the form

$$U(x,\epsilon) = U_1'(x) + \epsilon U_2'(x) + \epsilon^2 U_3'(x) + \ldots \tag{12}$$

where x may be a scalar or a vector argument. The partial sums are

$$s_n = \sum_{k=0}^{n-1} \epsilon^k U_{k+1}'(x) \tag{13}$$

and

$$s_{n+1} - s_n = \epsilon^n U_{n+1}'(x) \tag{14}$$

The condition (Eq. (2)] for the applicability of the extrapolation process yields

$$\epsilon \frac{U_{n+2}'(x)}{U_{n+1}'(x)} = \epsilon \ a(x), \quad (n \geq n_0) \tag{15}$$

This says specifically that, for any fixed x, the ratio of successive U's must be independent of n (to the desired accuracy) from some value n_0 onward. Thus the successive U's can no longer be chosen at will but must be the solution of the difference equation

$$U_{n+1}'(x) = a(x) \ U_n'(x) \tag{16}$$

which is

$$U_{n_0+p}'(x) = [a(x)]^p U_{n_0}'(x) \tag{17}$$

Under these assumptions, the tail of the perturbation series [Eq. (12)] can be written as a simple rational function and we have

$$U(x,\epsilon) = \sum_{k=0}^{n_0-1} \epsilon^k U_{k+1}'(x) + \epsilon^{n_0} \frac{U_{n_0+1}'(x)}{1 - \epsilon a(x)} \tag{18}$$

In some special cases it may turn out that $n_0 = 0$ and the power Series [Eq. (12)] is merely the expansion of

$$U(x,\epsilon) = \frac{U_1'(x)}{1 - \epsilon a(x)} \tag{19}$$

This is exactly the situation envisaged by Martin and Lomax. To determine when this method can be expected to work well, we merely put Eq. (19) for Eq. (12) into their "extended differential equation"

$$L(U) = (1 - \epsilon) F(U_0, x) + \epsilon F(U, x) \tag{20}$$

The result is

$$L\left[\frac{U_1'(x)}{1 - \epsilon \ a(x)}\right] = (1 - \epsilon) F(U_0, x) + \epsilon \ F\left[\frac{U_1'(x)}{1 - \epsilon \ a(x)}\right] \tag{21}$$

Here, because of the boundary conditions imposed on the terms of the series of Eq. (12) (see Eqs. (6) and (7) of the original paper[1]) $U_1'(x)$ must satisfy the boundary conditions imposed on the original $U(x)$, while $a(x)$ must satisfy the corresponding homogeneous (i.e., zero) boundary conditions. If two functions $U_1'(x)$ and $a(x)$ can be found which will satisfy Eq. (21) plus the indicated boundary conditions, then the method of Martin and Lomax will work perfectly. This is actually the case in their Example 1, as will now be shown.

The "One-Dimensional Example"

$$\left.\begin{aligned} &\frac{du}{dx}+u=\tfrac{1}{2}u^2 \qquad \text{in } 0\le x<\infty \\ &u(0)=1 \end{aligned}\right\} \tag{22}$$

employed by Martin and Lomax has the exact solution

$$u(x)=2e^{-x}/(1+e^{-x}) \tag{23}$$

It can be found by recognizing Eq. (22) to be a Riccati equation and solving by standard methods.

Whereas Martin and Lomax confine themselves to numerical comparisons of approximate results with the exact solution only at $x=1$, I shall show the analytical results of their methods as I consider these to be both interesting and instructive.

The method proposed by Martin and Lomax (referred to as "Method 2") consists of solving what they call the "extended form"

$$\left.\begin{aligned} &\frac{du}{dx}+u=(1-\epsilon)\tfrac{1}{2}u_0^2(x)+\epsilon\tfrac{1}{2}u^2(x) \\ &u(0)=1 \end{aligned}\right\} \tag{24}$$

by substituting into it the series [Eq. (12),] collecting coefficients of like powers of ϵ, and solving the resulting chain of differential equation with $u_1'(0)=1$, $u_n'(0)=0$ for $n>1$. The results is

$$\left.\begin{aligned} &u_0(x)=0 \\ &u_1'(x)=e^{-x} \\ &u_2'(x)=\tfrac{1}{2}e^{-x}(1-e^{-x}) \\ &u_3'(x)=\tfrac{1}{4}e^{-x}(1-e^{-x})^2 \end{aligned}\right\} \tag{25}$$

Aside from u_0, the first three terms are the first three terms of a geometric series the sum of which is nothing else but the exact solution, Eq. (23). This is why the numerical results presented by Martin and Lomax look so good.

But, so far, there is no guarantee that succeeding $u_n'(x)$ will continue to be successive terms of the same geometric series. That this is indeed true can be shown by inserting Eq. (19) into Eq. (24) and rearranging slightly

$$\left(\frac{du_1'}{dx}+u_1'\right)(1-\epsilon\, a(x))^{-1}+\epsilon\, u_1'(x)\left(\frac{da}{dx}-\frac{1}{2}u_1'(x)\right)$$

$$(1-\epsilon\, a(x))^{-2}=\tfrac{1}{2}(1-\epsilon)u_0^2(x) \tag{26}$$

An ad hoc solution can be obtained in three easy stages:

a) set $u_0(x)\equiv 0$, as Martin and Lomax do consistently;

b) solve $\dfrac{du_1'}{dx}+u_1'=0$ with the given boundary condition $u_1'(0)=1$; the result is

$$u_1'(x)=e^{-x} \tag{27}$$

c) solve $\dfrac{da}{dx}-\dfrac{1}{2}u_1'(x)=0$ with the corresponding boundary condition $a(0)=0$; the result is

$$a(x)=\tfrac{1}{2}(1-e^{-x}) \tag{28}$$

Putting these back into Eq. (19) gives

$$u(x,\epsilon)=\frac{e^{-x}}{1-\tfrac{1}{2}\epsilon(1-e^{-x})} \tag{29}$$

which, for $\epsilon=1$, reduces to

$$u(x)=\frac{e^{-x}}{1-\tfrac{1}{2}(1-e^{-x})}=\frac{2e^{-x}}{1+e^{-x}} \tag{30}$$

Thus I have shown that, for this particular example, the method proposed by Martin and Lomax does yield the exact solution.

It is good pedagogy, when introducing a new idea or technique, to demonstrate it on a "transparent example" wherein every step can be carried out explicitly and for which the final answer is known. Such is the case for the example previously discussed. The second example given by Martin and Lomax, flow over a biconvex airfoil, is, on the other hand, a real-life problem of the type for which their procedure was designed and provides a much more realistic test of its power.

In conclusion, I would like to point out that the Aitken/Shanks transform as used here is but the simplest possible example of the use of rational functions to extrapolate to the sum of an infinite series. This topic has been growing rapidly in recent years and goes under the name "Pade Approximation." See, for instance, the classic text by Wall[4] and the more recent volumes by Baker and Gammel[5] and by Baker.[6]

In another direction, extrapolation to the limit of a sequence can be effected using the sequence of transforms introduced by Shanks.[2] Their calculation has been systematized and made practical by Wynn[7] in his "epsilon algorithm." This he later extended[8] to vector sequences in his "vector epsilon algorithm." Convergence properties and applications of this algorithm have been studied extensively by Brezinski in his thesis[9] and many subsequent papers.

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